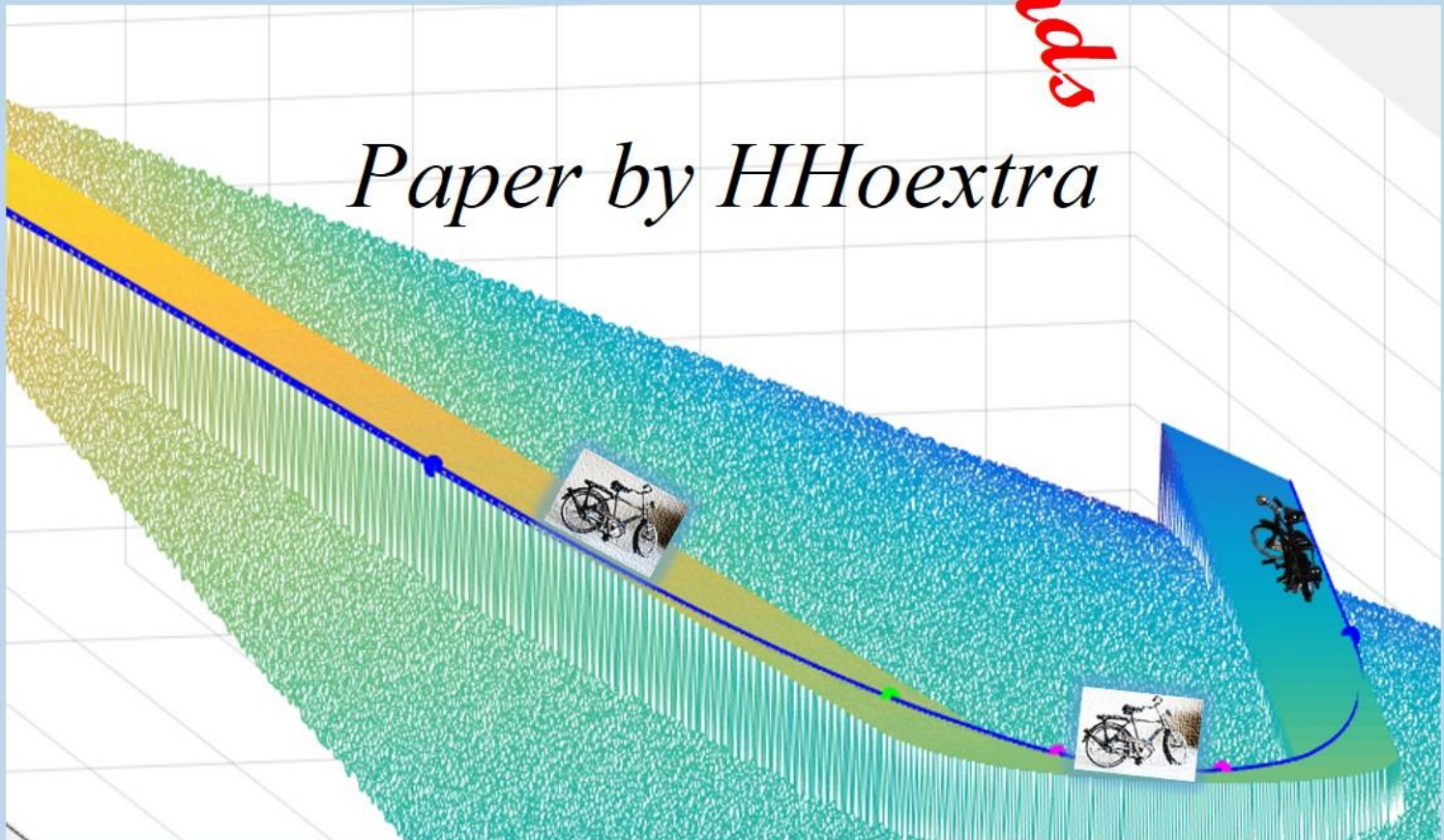


# *The Way of the Wheeler*

*in steep, descending bends*

*Paper by HHoextra*



## **Summary**

A presentation is given on theory and software for optimization of the travel time of a descending cyclist, taking sharp bends. Computational examples are presented and discussed as well as a few spectacular descents in recent classic tours (Milan-San Remo, Tour of Lombardia).



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# *The way of the wheeler in steep, descending bends*

## *Abstract*

The paper presents theory relevant for optimization of the travel time of a cyclist descending along bends in terms of all relevant parameters of both the cyclist-bike combination and the road structure. In addition, a scheme is presented to implement the above into software computing the ideal trajectory as well as the choreography of the actions to be taken by the cyclist. Computational results are given as an illustration to the above as well as a discussion on a few remarkable events in recent classic cycle tours, being the effect of the dropper seat post used by Matej Mohoric (Milan-San Remo, 2022) and the remarkably fast descent of Fausto Masnada from the Passa di Ganda (Tour of Lombardy, 2021).

## *I. Introduction*

Although racing downhill along bended roads in mountainous environments has been known for long to be decisive in a lot of cyclist races it was impossible for me to find –in a reasonable amount of time- scientific treatments of the topic via internet research. This together with recent observations of decisive descents via TV reports of classic tours were a motivation to start presented research. These tours are the classic Milan-San Remo of 2022, where winner Matej Mohoric took advantage of a so-called dropper seat post in bends, allowing him to lower his centre of gravity, if desired, and the classic Tour of Lombardy in 2021, where Fausto Masnada won about half a minute on his competitors in the descent of the Passa di Ganda. He, being a sun of the region and knowing the trajectory well, chose - going downhill along the many bends - the outer side of these bends where the road surfaces go up (contrary to his competitors who took the inner bend sides), thus improving the glance off conditions and so enabling a higher safe speed. He won a second place in that Tour of Lombardy in the same time as the winner Tadej Pogacar.

This paper presents a theory which takes into account all relevant parameters to optimize the travel time of riders in such situations. Examples will be given and the aforementioned observations are discussed using estimated parameters for the cyclist-bike combination and the road layout, leading to explanations, which are approximate –as one may expect-, for the events in those races.

The paper is organized as follows. In section II the theory, including details of its implementation into software, is presented, starting with an introduction of the relevant parameters for the bike-wheelsman combination and the used approach of the problem, in II.a. In section II.b the assumed parametric description of the bend layout is introduced and expressions related to that are presented. In section II.c all background expressions, relevant for the computational scheme are explained and discussed. The architecture of the optimizing software is given in section II.d.

In section III results of the above software are presented as an illustration to the method and also to get a better understanding of aforementioned events in the bicycle classics.

## *II. Theory and its implementation*

### *II.a. Parameters of the bike-cyclist combination and approach of the problem*

For the computation and minimization of the travel time of a cyclist going downhill along a road with a bend one needs to consider how the path can be optimized and how the speed distribution along the path depends on the various actions made by the cyclist, who in all cases

should never exceed the glance off speed. In this subsection the relevant parameters of the cyclist-bike combination are introduced as well as the approach for the optimization.

It is assumed that the cyclist-bike combination will have zero dimensions, i.e., it is assumed to have the size of a dot. Nevertheless using realistic parameters -as in this paper- to describe how different actions may influence the acceleration and so the speed while travelling on a bended road downhill all relevant effects that play a role can be incorporated with reasonable accuracy. A small error, however, introduced by the non-zero dimensions of the combination occurs in bends where –owing to the effect of the lean angle- the radius of curvature of the curve described by the centre of gravity (of the combination) will be smaller than that of the on-road trajectory of the bike (which in fact is used in the computations). If needed one could make an estimate of the true radius of curvature and adept the parameters of the road layout a bit to correct for the effect. We will make a few further remarks on this topic later on (section III).

The relevant parameters for the cyclist-bike combination are:

$$\begin{aligned}
P_{/mass}^{\max} &: \text{maximum power-mass ratio (in W/kg=m}^2/\text{s}^3\text{)}, \\
C_d &: \text{dragg coefficient (in 1/m)}, \\
a_s &: \text{acceleration due to slope} \\
a_b &: \text{maximum deceleration due to brake} \\
\alpha_L^{\max} &: \text{maximum of lean angle, } \alpha_L, \text{ and} \\
\alpha_p &: \text{limiting angle for pedalling.}
\end{aligned} \tag{1}$$

The quantity  $P_{/mass}^{\max}$  is the ratio of the maximum power the cyclist will use,  $P^{\max}$ , and the mass of the combination (cyclist-bike),  $m$ . As will be shown next that this quantity, together with the speed,  $v$ , determines that part of the acceleration coming from the work done by the cyclist, denoted by  $a_p$ . The power inserted by the cyclist will cause a changing kinetic energy as expressed by

$$P = 1/2 \left( \partial(mv^2) / \partial t \right)_p, \tag{2}$$

with the subscript  $P$  indicating that only the speed ( $v$ ) change caused by the cyclists power is considered, and  $t$  is the time. Then after differentiation we arrive at:

$$\left( \frac{1}{2} \frac{\partial mv^2}{\partial t} \right)_p = ma_p v \longrightarrow a_p = \frac{P}{mv} = \frac{P_{/mass}}{v}, \tag{3}$$

with equations 1 and 2.

The drag coefficient  $C_d$  is a consequence of the atmospheric air and is assumed to cause a deceleration described by

$$a_d = -C_d v^2. \tag{4}$$

Here the minus sign indicates that the acceleration diminishes the speed.  $C_d$  depends on both the nature of the combination bike-cyclist and the atmospheric pressure. The effect of wind, which may be large in particular in plane area, has been neglected in the present approach as it is usually small in mountainous area and for simplicity. Also, the relatively small effects of friction related to turning wheels and tires on a road surface are also neglected.

The acceleration due to the slope can be expressed by

$$a_s = g \sin \varphi, \tag{5}$$

with  $g(=9.81 \text{ m/s}^2)$  the gravitas and  $\varphi$  the slope angle as experienced by the cyclist, i.e.,  $\varphi$  is the slope angle of the road along the *exact* direction in which the cyclist moves.

The maximum deceleration due to using the brake can be written as

$$a_b = -\eta, \quad (6)$$

with  $\eta$  (typically in the range  $0.3g - 0.8g$ ) depending on among other things the type of brake and mass of the cyclist.

The maximum lean angle is the largest allowable angle of the cyclist with the normal to the local road surface in the direction perpendicular to that of the speed; for angles above  $\alpha_l^{\max}$  glance off would occur. It limits the allowable (for reasons of safety) speed of the cyclist and its value depends on the condition of the road as well as that of the tires.

The limiting angle for pedalling  $\alpha_p$  is the angle of the cyclist with the normal to the local road surface in the direction perpendicular to that of the speed at which the pedal at the inner side of the bend just touches the road surface. Here it is assumed that the cyclist takes the bend, if pedalling is impossible, with mentioned pedal up (i.e., in its highest position).

The *optimization method* used in this paper consists of optimizing the cyclist's trajectory (by minimizing the travel time), which is organized as follows:

1. The curved part of the trajectory is defined by a polynomial of which the coefficients are (being) optimized (using, e.g., an optimization routine) and of which the degree is gradually increased on optimizing until the outcome has converged with the desired accuracy. The curve is truncated in each computation at points where it runs parallel to the road sides at both sides of the bend (both for the first time as seen from the bend).
2. Straight sections, being parallel to the road sides are connected to the beginning and end points of the curved part of the trajectory, being continuations of it (guaranteeing transitions with continuous first order derivatives for the trajectory).
3. The travel time for each such trajectory computed assuming a set actions by the cyclist leading to the shortest time. For given trajectory and parameters such an optimized set of actions can be chosen in a unique way, as will be made plausible in sections II.c-d.

Its validity -in principle- as an optimization method leading to the fastest path can be made plausible by the following notions:

- It may be argued that the ideal trajectory *should* have beginning and end sections as mentioned under point 2 above, i.e., sufficiently far away from the bend, where the road is assumed to be straight and to have a constant slope, the ideal trajectory should be parallel to that road. One may expect that the resulting optimized trajectory in these straight sections run at the outer bend side (corresponding to a larger bend radius), but this is not imposed in the optimization procedure.
- In principle the polynomial degree can be so high that any *reasonable* curve, and so also the ideal one, could be approached up to any degree of accuracy. The word 'reasonable' has been used because only trajectories defined by a single valued function (for the chosen coordinate system, see sect. II.b) can be the outcome of the procedure (with polynomials to define the trajectory), which seems intuitively correct (although a proof of the latter seems not straightforward, and will not be given here).

But before presenting this method we first consider a parametric description of the type of bend structures that will be considered in the paper (although the optimization method is not limited to such a type).

## II.b. Parametric description of the bend layout

The perpendicular projection on the  $x$ - $y$  plane, which is chosen to be perpendicular to the gravity forces, of the bend layout is depicted in fig. 1. Next, sets of parameters that define both this layout and the height distribution will be given. As is seen in the figure: this projection of the layout is symmetric, unlike the structure as a whole owing to the presence of height variations, discussed later on. The road has a width,  $w$ , everywhere. It consists of two straight sections: the *entrance* (left hand side (lhs)) and *exit* (right hand side (rhs)) sections. These are connected by a bended structure, with inner band radius  $r_0$  and outer bend radius  $r_1 (= r_0 + w)$ , and an angular width of  $\pi - 2\theta_0$ . As a consequence of the latter: the angle between the straight sections is  $2\theta_0$ , which can be chosen in a range  $0 \lesssim 2\theta_0 < 180^\circ$ . The centreline of the road structure is presented in red (dashed) in fig. 1. It is indicated (blue arrows) that the bend is taken counter clockwise.

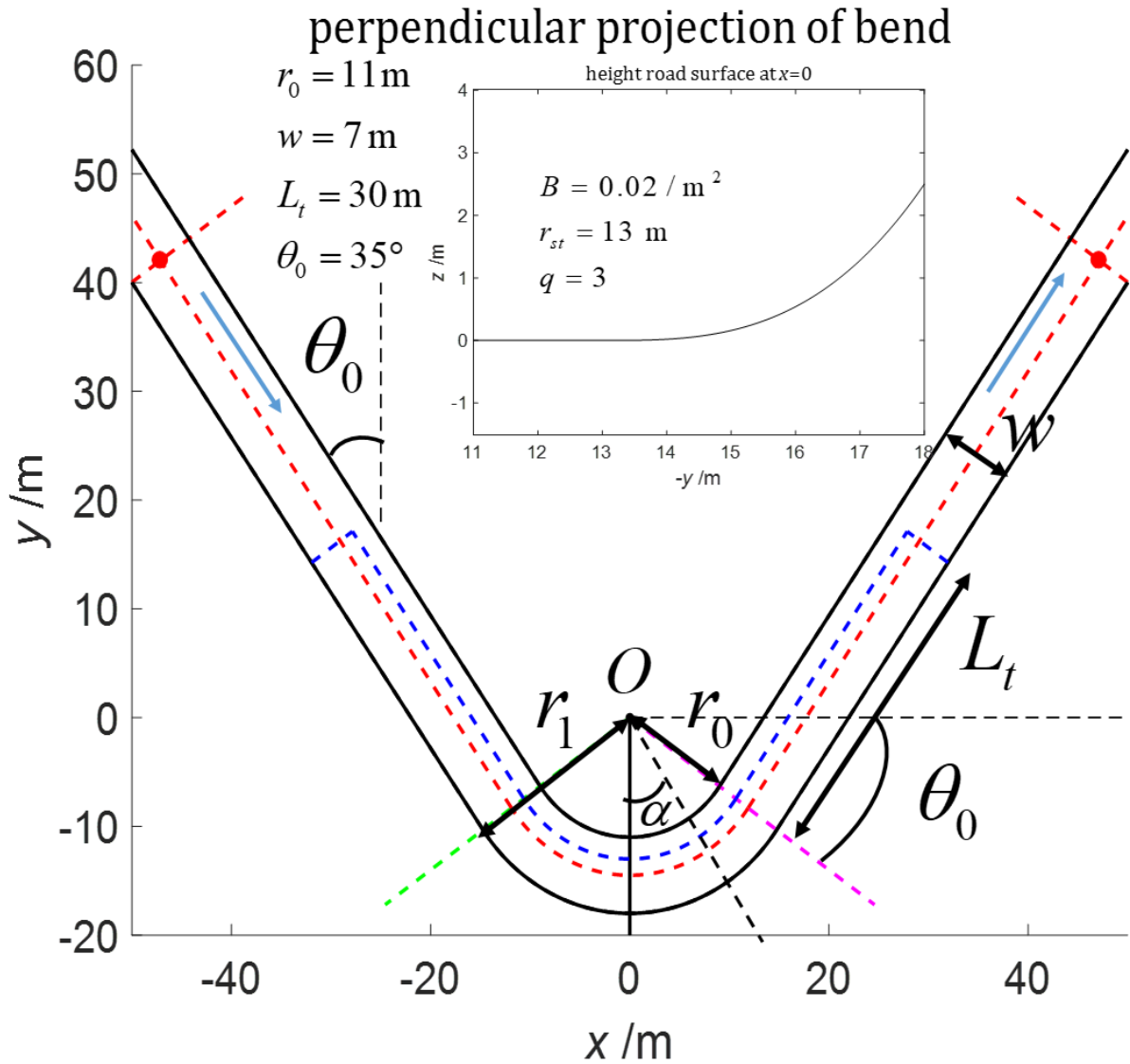


Figure 1. Example of the layout of considered bend structure for given parameters in the  $z=0$  plane. The blue arrows indicate the biking direction; the two red dots indicate the start and finish lines (as to the computations). The inset presents the height of the road surface, ascending towards the outer bend, in the  $x=0$  plane, for given parameters. The slope angle  $\varphi$  does not play any role in this figure.

The height distribution for the road surface,  $Z$ , consists of two contributions. One,  $Z_1(x, y)$  describes the nearly constant slope parallel to the road directions (downhill from the lhs to the rhs in fig. 1) and the second,  $Z_2(x, y)$ , describes possible height variations perpendicular to the road directions, which may be applied in the curved region to enable higher speeds (before glance off would occur). So, the height of the road surface is given by:

$$Z(x, y) = Z_1 + Z_2, \quad (7)$$

with  $(x, y)$  lying within the road boundaries as described above.

*Contribution  $Z_1(x, y)$ :* This quantity describes a slope being constant along the centreline mentioned earlier with a slope angle  $\varphi$ ; the value of  $Z_1$  is a constant in directions perpendicular to that of the road and we chose  $Z_1(0, -r_1 < y < -r_0) = 0$ . A presentation in formulae leads to the following for arbitrary points corresponding to the road surface defined by  $\mathbf{r}_{xy} \equiv (x; y)^t$ , with  $t$  indicating the transpose:

$$\begin{aligned} \text{bended part: } Z_1 &= -\alpha \bar{r} F \text{ if } \theta_0 - \pi/2 < \alpha < \pi/2 - \theta_0 \wedge r_0 < r_{xy} < r_1; \\ \partial Z_1 / \partial x &= -\bar{r} F y / r_{xy}^2, \quad \partial Z_1 / \partial y = \bar{r} F x / r_{xy}^2 \\ \bar{r} &\equiv (r_0 + r_1) / 2, \quad \alpha \equiv -\tan^{-1}(x / y), \quad r_{xy} \equiv |\mathbf{r}_{xy}|, \quad F \equiv \tan \varphi \\ \text{straight parts: } Z_1 &= \mp \{ (\pi/2 - \theta_0) \bar{r} F + D_{\pm} F \} \text{ if } r_0 < E_{\pm} < r_1 \wedge D_{\pm} > 0 \\ \partial Z_1 / \partial x &= -F \sin \theta_0; \quad \partial Z_1 / \partial y = \mp F \cos \theta_0 \\ D_{\pm} &\equiv \mathbf{r}_{xy} \cdot (\pm \sin \theta_0; \cos \theta_0)^t, \quad E_{\pm} \equiv \mathbf{r}_{xy} \cdot (\pm \cos \theta_0; -\sin \theta_0)^t; \text{ upper (lower) sign: rhs (lhs)} \end{aligned} \quad (8)$$

It is noted that in the above (and also below) the symbols  $D$  and  $E$  refer to inner products of  $\mathbf{r}_{xy}$  with unit vectors being parallel and perpendicular, respectively, to the side of the straight road sections. The upper (lower) subscripts refer to the rhs (lhs). Here, and in the rest of this paper, the bold font (e.g., for  $\mathbf{r}_{xy}$ ) is used to indicate the vectorial nature of considered quantity.

For later use (in computations for orientation of the local road surface) we also presented expressions for the partial derivatives of  $Z_1(x, y)$  in eq. 8.

*Contribution  $Z_2(x, y)$ :* This part takes care of variations perpendicular to the centreline enabling e.g. the introduction of a road surface ascending towards the outer side of the bend; this rise is assumed to start at a certain distance from the side of the road:  $r_{st}$  with  $r_0 < r_{st} < r_1$  (indicated by the blue dashed line in fig. 1). In the bended part  $Z_2(x, y)$  does not depend in the angle  $\alpha$ , introduced above, and its contribution decreases gradually (over a length  $L_t$ ) in the so called *transition sections* connected to the bend applied in the straight sections to avoid abrupt height changes. The part  $Z_2(x, y)$  is expressed as follows:

$$\begin{aligned} \text{bended part: } Z_2 &= B h^q \text{ if } r_{st} < r_{xy} < r_1 \wedge \theta_0 - \pi/2 < \alpha < \pi/2 - \theta_0; \\ \partial Z_2 / \partial x &= q B h^{q-1} x / r_{xy}, \quad \partial Z_2 / \partial y = q B h^{q-1} y / r_{xy} \\ h &\equiv r_{xy} - r_{st}, \quad q = 1..3, \quad \alpha = -\tan^{-1}(x / y) \\ \text{straight parts: } Z_2 &= B (E_{\pm} - r_{st})^q (L_t - D_{\pm}) / L_t \text{ if } r_{st} < E_{\pm} < r_1 \wedge 0 < D_{\pm} < L_t \\ \partial Z_2 / \partial x &= \pm \cos \theta_0 q B (E_{\pm} - r_{st})^{q-1} (L_t - D_{\pm}) / L_t \mp B (E_{\pm} - r_{st})^q \sin \theta_0 / L_t \\ \partial Z_2 / \partial y &= -\sin \theta_0 q B (E_{\pm} - r_{st})^{q-1} (L_t - D_{\pm}) / L_t - B (E_{\pm} - r_{st})^q \cos \theta_0 / L_t \end{aligned} \quad (9)$$

For later use we also presented expressions for the partial derivatives of  $Z_2(x, y)$  in eq. 9.

## II.c. Background expressions for the computational scheme

### Polynomial expansion of the trajectory

As explained in section II.a an important part of the trajectory of the cyclist is the curved part of it, defined by a polynomial of the form

$$y^c = \sum_{l=0}^P A_l x^l. \quad (10)$$

Together with the height function discussed in section II.b this defines the corresponding 3D curve which will be discretized for the computations, leading to a set of, say,  $N$  points denoted by  $\mathbf{r}_{1:N_c}^c$ . The speed in these points will be denoted by  $\mathbf{v}_{1:N_c}^c$ .

### The stationary speed

From the quantities and expressions (see eqs. 1-6) in section II.a an expression for the total acceleration of the cyclist in a direction parallel to the speed,  $a$ , can be derived, being

$$a = P_{/mass} / v + g \sin \varphi - \gamma \eta - C_d v^2 \equiv \chi_1 / v + \chi_2 + \chi_3 v^2, \quad (11)$$

with  $\gamma \in (0,1)$  the fraction of the braking deceleration being used and assuming a slope angle  $\varphi$ . The expression after the second equality sign has been introduced to simplify the notation, for later use. If the fraction  $\gamma$  is zero and at constant slope the corresponding *stationary speed*, say,  $v_s$ , for which  $a = 0$ , can be solved from a cubic equation with solution [1]:

$$v_s = 2W \cos\left(\cos^{-1}[3P_{/mass} / (2Wg \sin \varphi)] / 3\right), \quad W \equiv \sqrt{g \sin \varphi} / \sqrt{3C_d}. \quad (12)$$

This will be his speed if the cyclist is sufficiently far away from the bend, but is (still) experiencing a constant gravitational acceleration ( $g \sin \varphi$ ).

### Glance off speed, $v_x$ , and other quantities of the curved part of the trajectory

This subsection starts with considering expressions for a local coordinate system at a certain point along the cyclists' trajectory. Thereafter, the accelerations felt by the cyclist are identified and a derivation of expressions of these using local coordinates is given, leading eventually to the desired expressions for the glance off speed,  $v_x$ , corresponding to a given maximum lean angle  $\alpha_L^{\max}$ .

First, we present an analysis of important vectors describing the local road surface orientation for discretization points along the curved trajectory, being the unit vectors parallel to the speed, say,  $\mathbf{e}_v$ , the unit vector normal to the surface, say,  $\mathbf{e}_n$ , and the one perpendicular to these two:  $\mathbf{e}_p$ . The first two follow directly from

$$\mathbf{e}_v = \frac{\mathbf{d}}{|\mathbf{d}|}, \quad \mathbf{d} \equiv \begin{pmatrix} 1 \\ y'^c \\ z' + (\partial z / \partial y) y'^c \end{pmatrix}, \quad \mathbf{e}_n = \frac{\mathbf{h}}{|\mathbf{h}|}, \quad \mathbf{h} \equiv \begin{pmatrix} z' \\ \partial z / \partial y \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \partial z / \partial y \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ z' \end{pmatrix}, \quad (13)$$

where the prime denotes differentiation with respect to  $x$ . In the above it is used that  $\mathbf{e}_n$  is the normalized result of the outer product of two in-surface vectors, its direction is chosen such that its  $z$ -component is negative (such that the main direction of  $\mathbf{e}_n$  is along  $-z$ ). It is noted that  $(\mathbf{e}_n \cdot \mathbf{e}_v) = 0$ , as expected, because the cyclist moves parallel to the road surface. The expression for vector  $\mathbf{e}_p$ , pointing away from the cyclist to his rhs, now follows with

$$\mathbf{e}_p = \mathbf{e}_n \times \mathbf{e}_v. \quad (14)$$

It is well known that the lean angle of the cyclist along curve trajectories should not be too large in order to prevent glance off. This angle depends among other things on the speed of the cyclist and on the radius of curvature of its path. We next will derive expressions to compute the *glance off speed*,  $v_{x,m}$ ,  $m = 1 - N$ , (which is crucial for the choreography of the cyclist) corresponding to the maximum lean angle  $\alpha_L^{\max}$ .

The surface underneath the cyclist will (naturally) be parallel to  $e_v$  and the centrifugal acceleration  $a_c$  should be perpendicular to  $e_v$  (as follows from basic theory). The situation is illustrated in figure 2 representing relevant unit vectors in a plane perpendicular to  $e_v$ , which is assumed to point out of the paper. The vector  $a_{g\perp v}$  is the component of the gravitational acceleration felt by the cyclist-bike combination ( $=a_g$ ) perpendicular to its speed and can be computed from:

$$a_{g\perp v} \equiv a_g - (a_g \cdot e_v)e_v (= a_g - a_{g\parallel v}), \quad (15)$$

as is seen easily. It is noted that the acceleration  $a_{g\parallel v}$  is accounted for in the calculation of speed changes. The other force acting on the combination at, say, point  $r_2^c$  is related to the centrifugal acceleration,  $a_c$  which is given by

$$a_c = v_2^c{}^2 e_c / R_2, \quad (16)$$

with  $e_c$  the corresponding unit vector and  $R_2$  the radius of curvature of the path of the combination and  $v_2^c$  is the speed. These two quantities can be computed as follows:

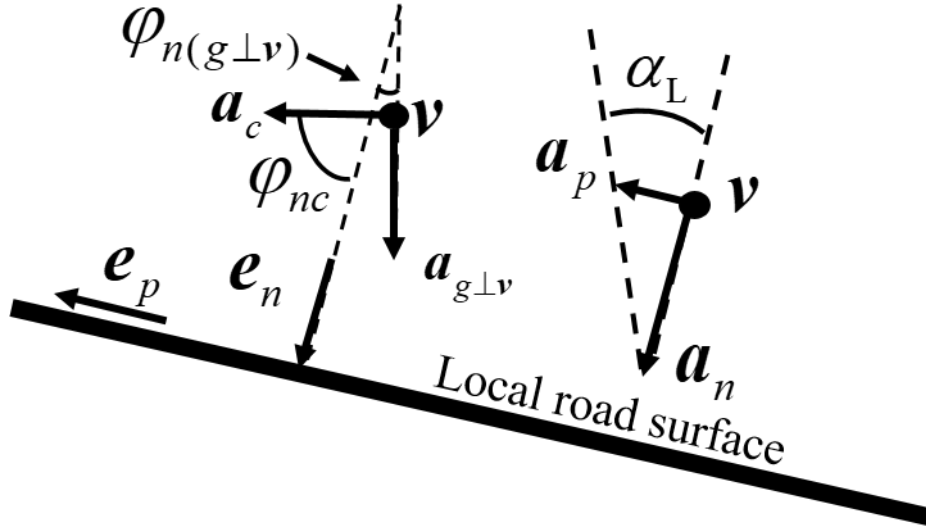


Figure 2. Acceleration vectors and local unit vectors for the surface in a plane perpendicular to the direction of the speed (vector  $v$ , pointing out of the paper), relevant for the glance off speed.

The unit vector perpendicular to the plane defined by the points  $r_1^c, r_2^c, r_3^c$ , say,  $e_{2\perp}$ , can be calculated from

$$e_{2\perp} = r_{32}^c \times r_{21}^c / |r_{32}^c \times r_{21}^c|; r_{32}^c \equiv r_3^c - r_2^c; r_{21}^c \equiv r_2^c - r_1^c, \quad (17)$$

with the notification that the curvature is zero if the first outer product is zero (and so the considered point would need no further consideration as to centrifugal forces). The following vector expressions, defining the collection of points of the two perpendicular bisectors of the segments between points  $r_1^c, r_2^c$  and  $r_2^c, r_3^c$  in considered plane, are defined by

$$w_{32} = (r_3^c + r_2^c) / 2 + \lambda r_{32}^c \times e_{2\perp}, w_{21} = (r_1^c + r_2^c) / 2 + \mu r_{21}^c \times e_{2\perp}, \quad (18)$$

assuming variable (real) factors  $\lambda$  and  $\mu$ . They both lie in the plane through considered three points and their crossing point defines the corresponding centre of curvature. The expression for that crossing point can be found by demanding  $\mathbf{w}_{32} = \mathbf{w}_{21}$ , leading to three equations from which e.g.  $\lambda$  can be solved straightforwardly. From these one gets the solutions, say,  $\lambda_0$  and  $\mathbf{w}_{32,0}$ , and so expressions for the vector connecting points  $\mathbf{w}_{32,0}$  and  $\mathbf{r}_2^c$  and the radius of curvature at point  $\mathbf{r}_2^c$ :

$$\mathbf{e}_c = (\mathbf{r}_2^c - \mathbf{w}_{32,0}) / R_2, \quad R_2 = |\mathbf{r}_2^c - \mathbf{w}_{32,0}|. \quad (19)$$

This procedure, and its follow up as given below, has to be repeated for the points numbered  $3 : N_c - 1$ .

It is noted that three discretized points are used to solve for  $R_{2:N_c-1}$  and that, if one of these points lies across a boundary between two sections (like bended and straight parts) inaccuracies may arise in the computed results. In that case it is recommended to use results of neighbouring points (in the same section); this has been done for all the presented results in this paper. Similar things can be done to get approximate values for  $R_1$  and  $R_{N_c}$ , via extrapolation or simply by putting, e.g.,  $R_1 = R_2$  etc..

An alternative to the above approach would have been to compute these radii of curvature and  $\mathbf{e}_c$  via differential calculus, but that is much more involved (and leads to virtually the same computational results, as we have found).

Next we need to consider what the accelerations mentioned in eqs. 15 and 16 mean for the lean angle,  $\alpha_L$ , which is expressed in terms of the accelerations normal and parallel to the local surface, denoted by respectively  $\mathbf{a}_n$  and  $\mathbf{a}_p$ , vectors with lengths  $a_n$  and  $a_p$ , as follows:

$$\tan \alpha_L = a_p / a_n, \quad a_p \equiv \mathbf{e}_p \cdot \mathbf{a}_p, \quad \mathbf{a}_n \equiv \mathbf{e}_n \cdot \mathbf{a}_n, \quad (20)$$

with  $a_p > 0$  if  $\mathbf{a}_p$  points to the rhs of the cyclist. The sum of these accelerations is the total acceleration in the plane perpendicular to  $\mathbf{v}$  and, so, it should obey the equality

$$\mathbf{a}_{g \perp \mathbf{v}} + \mathbf{a}_c = \mathbf{a}_p + \mathbf{a}_n \longrightarrow \mathbf{a}_g + \mathbf{a}_c - \mathbf{a}_{g \parallel \mathbf{v}} = \mathbf{a}_p + \mathbf{a}_n, \quad (21)$$

where eq. 15 was used for the second equality. Strictly speaking one should also check whether  $a_n > 0$ , i.e., the cyclist should keep contact with the road surface, but such a thing will certainly be the case under not very extreme circumstances as considered here.

From eq. 21 we arrive at the following (after taking the dot product with  $\mathbf{e}_n$  and  $\mathbf{e}_p$ , respectively):

$$\begin{aligned} a_g A_{gn} + a_c A_{cn} &= a_n, \quad A_{gn} \equiv \mathbf{e}_g \cdot \mathbf{e}_n (= -e_{n,z}), \quad A_{cn} \equiv \mathbf{e}_c \cdot \mathbf{e}_n = \cos \varphi_{nc}, \\ a_g A_{gp} + a_c A_{cp} &= a_p, \quad A_{gp} \equiv \mathbf{e}_g \cdot \mathbf{e}_p (= -e_{p,z}), \quad A_{cp} \equiv \mathbf{e}_c \cdot \mathbf{e}_p = \cos(90^\circ - \varphi_{nc}) = -\sin \varphi_{nc}, \end{aligned} \quad (22)$$

with the angle  $\varphi_{nc}$  depicted in fig. 2. In the above we used that  $\mathbf{e}_g$  is the unit vector along  $-\mathbf{z}$  and that  $\mathbf{e}_{g \parallel \mathbf{v}}$  is perpendicular to the plane spanned by  $\mathbf{e}_n$  and  $\mathbf{e}_p$  (and so, for example  $\mathbf{a}_{g \parallel \mathbf{v}} \cdot \mathbf{e}_n = 0$ ). Then, the above quantities can be related to the lean angle  $\alpha_L$  via the following expressions:

$$T = \frac{a_p}{a_n} = \frac{A_{gp} + \gamma A_{cp}}{A_{gn} + \gamma A_{cn}}, \quad \gamma \equiv a_c / g, \quad T \equiv \tan \alpha_L, \quad a_c = v^2 / R. \quad (23)$$

One may be tempted to compute  $v_x$  from eq. 23 by solving for  $\gamma_{\max} = \gamma(\alpha_L^{\max})$ , which would work well in simple situations (as to slopes and trajectory), but not always. Before describing how to solve for the glance off speed in a safe way eq. 23 will first be rewritten. To that end we start with factorising the quantity  $A_{gn}$  via a decomposition of  $\mathbf{e}_g (= (0, 0, -1)^t)$  into mutually perpendicular unit vectors, one along  $\mathbf{v}$  (being  $\mathbf{e}_v$ ) and one perpendicular to  $\mathbf{v}$ , say,  $\mathbf{e}_{g\perp v}$ :

$$\mathbf{e}_g = (\mathbf{e}_g \cdot \mathbf{e}_{g\perp v})\mathbf{e}_{g\perp v} + (\mathbf{e}_g \cdot \mathbf{e}_v)\mathbf{e}_v, \quad \mathbf{e}_{g\perp v} = \mathbf{a}_{g\perp v} / |\mathbf{a}_{g\perp v}|, \quad (24)$$

with  $\mathbf{a}_{g\perp v}$  given by eq. 15. Then, substituting eq. 24 into the expression for  $A_{gn}$  (see eq. 22) it follows, using  $(\mathbf{e}_n \cdot \mathbf{e}_v) = 0$  (see note below eq. 13):

$$A_{gn} = (\mathbf{e}_g \cdot \mathbf{e}_{g\perp v})(\mathbf{e}_{g\perp v} \cdot \mathbf{e}_n) \equiv \cos \varphi_{g(g\perp v)} \cos \varphi_{n(g\perp v)} \quad (25)$$

$$\text{similarly: } A_{gp} = -\cos \varphi_{g(g\perp v)} \sin \varphi_{n(g\perp v)}$$

with  $\varphi_{n(g\perp v)}$  the angle between  $\mathbf{e}_n$  and  $\mathbf{e}_{g\perp v}$  (see Fig. 2), and  $\varphi_{g(g\perp v)}$  the angle between  $\mathbf{e}_g$  and  $\mathbf{e}_{g\perp v}$  (and so:  $\cos \varphi_{g(g\perp v)} = -e_{g\perp v, z}$ ). Now eq. 23 can be rewritten into:

$$\gamma = \frac{TA_{gn} - A_{gp}}{A_{cp} - TA_{cn}} = \cos \varphi_{g(g\perp v)} \frac{\sin(\alpha_L + \varphi_{n(g\perp v)})}{\sin(\varphi_{nc} - \alpha_L)}, \quad (26)$$

where the final expression contains (fortunately) only a few independent variables.

It may reasonably be assumed that the quantities in the numerator of the rhs expression of eq. 26 will be positive (typically:  $\alpha_L^{\max} \approx 30^\circ - 50^\circ$ ,  $\varphi_{nc} \approx 80^\circ - 100^\circ$ ). Then, it is seen that  $\gamma$ , and so  $v$ , may become very large at  $\varphi_{nc} \approx \alpha_L$  and  $\gamma_{\max}$  will be negative if  $\varphi_{nc} < \alpha_L^{\max}$ , implying imaginary values for  $v_x$  in such a case, which is not the type of solution we are looking for. But, if  $\varphi_{nc} < \alpha_L^{\max}$  then  $\gamma$  will become very large at lower values of  $\alpha_L$ , and  $v_x$  can be chosen to be very large for considered point along the curved trajectory with still  $\alpha_L \ll \alpha_L^{\max}$ . So, a simple, but well working suggestion to solve for  $\gamma_{\max}$  (and so  $v_x$ ) is the following:

1. In a first step it is checked with eq. 23 (at each location in the curved part of the trajectory) whether the lean angle  $\alpha_L$  has a sufficiently small value (i.e. whether  $\alpha_L < \alpha_L^{\max}$ ) at a just unreachable speed, say,  $v_{\text{lim}}$ , with, for example,

$$v_{\text{lim}} \equiv 1.05v_s, \quad (27)$$

with  $v_s$  introduced by eq. 12. If so, we put for considered location  $v_x = v_{\text{lim}}$ .

2. If not, eq. 26, with  $\alpha_L = \alpha_L^{\max}$ , is used to compute the sought value for  $v_x$ .

This procedure has shown to lead to real and correct values for  $v_x$  and has been used in all the presented computational results in section III. All the resulting solutions showed indeed that nowhere the value of  $\alpha_L^{\max}$  was exceeded (see e.g. graphs in sect. III).

In a similar way one may derive an expression for the pedalling speed limit,  $v_p$ , with now using  $\alpha_p$  (the limiting angle for pedalling introduced in eq. 1) instead of  $\alpha_L^{\max}$ . Alternatively, one may also try to solve for  $v_p$  directly from eq. 26 as usually the following holds:  $\alpha_p \ll \alpha_L^{\max}$ , i.e., the conditions are here much more relaxed for  $\alpha_p$  compared to that for  $\alpha_L^{\max}$  (usually).

In our software all values of both  $v_x$  and  $v_p$  exceeding  $v_{lim}$  are put equal to  $v_{lim}$ , for compacter pictures with easily recognizable features (see e.g. section III).

### *Speed and travel time evaluation*

In order to compute efficiently the time needed for one small step along the curve (say, from  $s_1$  to  $s_2$ , with  $\Delta s = s_2 - s_1$  being sufficiently small) and also to calculate the new speed after that step, the following scheme is used taking also into account (for higher accuracy) that the speed is changing on the way along the (small) step  $\Delta s$ . Doing so we arrive, using a Taylor expansion, at:

$$v_2 = v_1 + \left. \frac{\partial v}{\partial s} \right|_{s_1} \Delta s + \left. \frac{\partial^2 v}{2 \partial s^2} \right|_{s_1} \Delta s^2. \quad (28)$$

The derivatives in there can be rewritten using as a starting point the conservation of energy law:

$$\Delta(v^2) / 2 = a \Delta s \xrightarrow{\text{whence}} \partial v / \partial s = a / v. \quad (29)$$

Next, using the second equality of eq. 11, we arrive at

$$\frac{\partial v}{\partial s} = \chi_1 / v^2 + \chi_2 / v + \chi_3 v; \quad \frac{\partial^2 v}{\partial s^2} = \left( -2\chi_1 / v^3 - \chi_2 / v^2 + \chi_3 \right) \frac{\partial v}{\partial s}, \quad (30)$$

which can be used to evaluate  $v_2$  with eq. 28. The elapsed time during step  $\Delta s$ ,  $\Delta t$  can be approximately computed as follows

$$\Delta t = \int_{s_1}^{s_2} (1/v) ds = (1/v)_{\text{average}} \Delta s \approx (v_1 + v_2) \Delta s / (2v_1 v_2), \quad (1/v)_{\text{average}} \equiv (1/v_1 + 1/v_2) / 2. \quad (31)$$

It is noted that the above computational scheme (eqs. 28-31) can equally well be used for calculation of time reversed cycling, which is needed in some cases, as discussed hereafter. Naturally both  $\Delta s$  and  $\Delta t$  are negative in such cases.

### *The safe speed, $V_x$*

Ideally, the optimum choice for the cyclist's speed might seem to be the choice  $v = v_x$ , with  $v_x$  computed as discussed below eq. 26, but  $v_x$  may change too fast at certain positions along the path compared to what is possible with given maximum braking deceleration, introduced in eq. 6. Before such positions are reached the cyclist should already decelerate in order to avoid glance off accidents. Below we will show how such points, or regions, along the path are identified, and also how to cope with these by adopting in time a sufficiently low velocity at earlier parts of the trajectory, say the *safe speed*  $V_x$ , with (of course) everywhere  $V_x \leq v_x$ .

But, before doing so we introduce a way to identify numerically the critical parts of the trajectory in case the cyclist would travel with speed  $v_x$ . The deceleration required in between neighbouring positions of the discretized curve, say,  $s_1$  to  $s_2$ , with values for the glance off speed  $v_x$  labelled correspondingly, follows with eqs. 28 and 29:

$$\Delta v_x = \int_{s_1}^{s_2} \partial v_x / \partial s ds = \int_{s_1}^{s_2} a_x / v_x ds = a_x (1/v_x)_{\text{average}} \Delta s \rightarrow a_x = \Delta v_x / \left[ (1/v_x)_{\text{average}} \Delta s \right], \quad (32)$$

$$\text{with } (1/v_x)_{\text{average}} \approx (v_{x,1} + v_{x,2}) / (2v_{x,1} v_{x,2}); \quad \Delta v_x \equiv v_{x,2} - v_{x,1},$$

with  $a_x$  the acceleration for a cyclist with speed  $v_x$ . If the cyclist would have a speed  $v_x$  it should be avoided that  $a_x$  is smaller (larger in absolute sense) than the minimum acceleration that can be achieved with given parameters, say,  $a_{\min}$ . So, one should have along the full curve (assuming speed  $v_x$ )

$$a_x > a_{\min} = a_{g\parallel v} - \eta - C_d v_x^2; \quad (a_{g\parallel v} \equiv |\mathbf{a}_{g\parallel v}| = g \sin \varphi), \quad (33)$$

with  $a_{\min}$  corresponding to zero input power and a full brake (acceleration  $-\eta$ ) at a speed of the (local) value of  $v_x$ . If it happens that for a certain part of the trajectory, i.e. a number of discretization points, the following inequality holds

$$a_x - (a_{g\parallel v} - \eta - C_d v_x^2) < 0, \quad (34)$$

the cyclist has to adopt a lower speed, the safe speed  $V_x$ , while running up. The way to compute the latter will be discussed next.

At first, the first point beyond the *highest* value of  $x$  for which eq. 34 holds is selected, say, this is at, say,  $\mathbf{r}_M^c$ . For this point and the points further on along the trajectory the speed  $v_x$  is apparently a safe speed and we may put  $V_{x,M:N_c} = v_{x,M:N_c}$ . Next, starting from that point  $\mathbf{r}_M^c$  on, the time reversed path of the cyclist is evaluated assuming a speed in  $\mathbf{r}_M^c$  of  $V_{x,M}$  and a maximum braking deceleration (see eq. 6). The required formulae for that are presented in eqs. 28-30, but now (going backwards!) one should take  $\Delta s$  to be negative. In these backwards calculations each newly calculated speed is the safe speed for considered point (which should be stored in array  $V_x$ , with  $V_x \leq v_x$ ). As a consequence: arriving at that point having that speed, the (forward going) cyclist can indeed safely follow his way as long as his speed does not exceed the safe speed.

These backwards computations are truncated if a newly computed  $V_x$  value is larger than the  $v_x$  value in that point, say, at  $\mathbf{r}_p^c$ , or if the first point of the curved path is reached (at  $\mathbf{r}_1^c$ ). If the latter happens the safe speed along considered trajectory (between points  $\mathbf{r}_M^c$  and  $\mathbf{r}_1^c$ ) is known and stored for later use (see below). If the first thing happens, i.e., in point  $P$  the  $V_x$  value is larger than the  $v_x$  value, the safe speed in that point is given by  $V_{x,P} = v_{x,P}$ , because the cyclist's speed may never exceed the glance off speed. Next, the rest of the trajectory (between points  $\mathbf{r}_{p-1}^c$  and  $\mathbf{r}_1^c$ ) is investigated in the same way as described above and  $V_x$  values are computed in the same way as above, if required. If inequality 34 does not hold for any point in that area we may put there  $V_x = v_x$  in considered range. This procedure is repeated, if required, until a safe speed  $V_x$  is obtained for the complete curved part of the trajectory.

If at the beginning of the curved part either  $V_{x,1} < v_{\lim}$  or  $\mathbf{r}_1^c$  is not in a region having a constant slope (i.e., the point is not lying before the lhs transition element) the backwards computation should continue into the straight part of the trajectory at the lhs after having introduced a discretization over there, with coordinates of the trajectory:  $\mathbf{r}_m^b$ ,  $m = 1 - N_b$ , with  $\mathbf{r}_1^b$  on the starting line. The backwards calculations should then continue until either a point  $p$  is reached for which both  $V_{x,p} > v_{\lim}$  and the slope is constant (i.e.,  $\mathbf{r}_p^b$  lies before the start of the transition section) or until the first point of the trajectory at  $\mathbf{r}_1^b$  (which should lie before the lhs transition section). How to calculate the *excess time* (i.e., the extra time owing to the need to have a speed

at the start of the computations below the stationary speed,  $v_s$ ) if the latter happens to be the case is explained in section II.d.

If there is *not* any discretization point along the full curved part of the trajectory for which inequality 34 holds(which could happen for gentle slopes or bends) the actual calculation of the travelling time can start using  $V_x = v_x$  as a safe speed for the full trajectory.

It is further noted that the cyclist will generally not always manage to reach the local speed  $V_x$  at each point along the trajectory. He may of course nowhere exceed the local safe speed,  $V_x$ .

### *Keeping on track*

There is another point of concern that we did not mention so far: the trajectory should lie on the road and not go through its boundaries. To that end a penalty factor is introduced for the travel time which becomes active in case the path leaves the road, forcing the optimization algorithm to keep right track. This means, e.g., that for each step along the track the elapsed time as computed like given in eq. 31 is multiplied with a factor,  $f_{\text{retard}}$ , as follows:

$$\Delta t_{\text{acc}} = f_{\text{retard}} \Delta t. \quad (35)$$

The result,  $\Delta t_{\text{acc}}$ , is used in the calculations of the total extra time. There is a lot of freedom in choosing the factor  $f_{\text{retard}}$ ; important is that it is unity if considered step is fully on the road and that it increases with the distance between considered step and the road side if it is off road. We used as a criterion for the latter the position of the starting point of considered step projected in the  $x$ - $y$  plane (say  $(x_n, y_n)$ ) as follows:

$$f_{\text{retard}} = \begin{cases} 1; & d < w/2 \\ 10e^{d-w/2}; & d, w \text{ in hm} \end{cases}, \quad (36)$$

with  $d$  defined as the distance between considered point and the heart line of the road ( $w$  is the width of the road). So correction, i.e.  $f_{\text{retard}} > 1$ , will only occur if considered point is not lying on the road. The distance  $d$  can be computed straightforwardly, e.g., for the straight road segment on the rhs, defined by  $-\theta_0 < \tan^{-1}(y_n/x_n) < \pi/2$ ,  $x_n > 0$ , with  $(x_n, y_n)$  being the coordinates in the  $z=0$  plane, we find:

$$d = |y_n \sin \theta - x_n \cos \theta + \bar{r}|, \quad \bar{r} \equiv (r_0 + r_1)/2. \quad (37)$$

Similar expressions can be found for the straight segment on the lhs (change  $x_n \rightarrow -x_n$ ) and the bend section.

We are now ready to use the above ingredients for a computational scheme, introduced next.

### *II.d. The computational optimizing scheme*

The optimization software contains among other things of a call to a minimization routine which calls, while optimizing, a module that computes the *effective time* needed to take the bend. With the effective time the following is meant: the time needed to travel from the starting line to the finish line to which is added the *excess time* in case the cyclist has to use the brake before the starting line and/or did not yet have his stationary speed,  $v_s$ , at the finish line. The starting line is defined by a line (in the  $z=0$  plane) through  $x = x_{\text{min}}$  and perpendicular to the road direction; a similar thing holds for the finish line at  $x = x_{\text{max}}$ , with usually  $x_{\text{min}} = -x_{\text{max}}$ . Both lines are indicated in Fig. 1 (red dashed lines) and the  $x$ -axis corresponds then to the interval  $(x_{\text{min}}, x_{\text{max}})$ . To obtain a travel time independent of the choice for the starting and finish line positions we will compute the so-called *extra time*,  $\tau$ , needed to round the bend, which results from the difference between above *effective time* and the time needed to travel along the same

distance on a straight road having the same slope as considered structure (and so a speed,  $v_s$ , as introduced in eq. 12). For mentioned ‘same distance’ it is assumed that the cyclist follows the heart line in the bend (corresponding to a red dashed line in fig. 1).

The module to compute the *effective time* requires as an input the following:

1. values for the parameters given in eq. 1,
2. all structural parameters introduced in section II.b, and the computed corresponding stationary speed  $v_s$  (see eq. 12)
3. polynomial coefficients  $A_{0-M}$  (see eq. II.b) with  $M$  typically in the range 2 to ~6, defining, together with parameters for the road surface, the trajectory
4. arrays to calculate *excess* time losses in begin and end sections, in case the computations begin and/or end with a speed smaller than  $v_s$ , as explained in more detail below, in point 8.
5. a value for the step size for the discretization of the cyclist’s path in the  $x - y$  plane,  $\Delta s_{xy}$  and values for the size of the computational window along the  $x$ -axis, say,  $x_{\max}$  with  $-x_{\max} < x < x_{\max}$ . Its value should be chosen sufficiently far away from the bend section such that both the curved section is lying fully within the window as well as the two (lhs and rhs) transition sections. Normally a value for  $x_{\max}$  of around 30 m is sufficient, if required a bit larger value is chosen.

The module is organized along the following lines:

1. Determination of the positions in the  $z = 0$  plane closest to the bend where the polynomial curve is parallel to the road sides left (say at  $\mathbf{r}_{xy}^{c,L}$ ) and right (say at  $\mathbf{r}_{xy}^{c,R}$ ) of the bend by looking for zeros of  $y'^c(x) \mp \cot \theta_0 = 0$ , (+ for lhs, - for rhs) at positions being closest to the bending section with e.g. a zero finding module (as explained at the end of section II.a).
2. The discretization points of the curved part of the path in the  $x$ - $y$  plane are computed such that the distances between neighbouring points are all around  $\Delta s_{xy}$ , starting at  $\mathbf{r}_{xy,2}^c \equiv \mathbf{r}_{xy}^{c,L}$  (the reason to choose  $\mathbf{r}_{xy}^{c,L}$  to be the second point will be explained below). Each next point is then computed using

$$x_{p+1}^c = x_p^c + \Delta s_{xy} / \sqrt{1 + y_p'^c{}^2}, \quad (38) \quad \text{with}$$

$(x_p^c, y_p^c) \equiv \mathbf{r}_{xy,p}^c$ ;  $y_p^c$  and  $y_p'^c$  are evaluated using the corresponding polynomial (see eq. 10) and the one defining the derivative of it. This computation stops at  $x_{N_c-1}^c$  being the first value of  $x_{p+1}^c$  larger than  $x^{c,R}$  (being the  $x$ -coordinate of  $\mathbf{r}_{xy}^{c,R}$ ). The points  $(x_1^c, y_1^c)$  and  $(x_{N_c}^c, y_{N_c}^c)$  are extensions of the curve lying on a distance  $\Delta s_{z=0}$  away of their respective neighbouring points, defining the first points of the paths in the straight sections parallel to the road sides, left and right of the bend. Their introduction is needed to enable the calculation of the glance off speed  $v_x$  along the curved path (discussed next) at the begin and end points (labelled 2 and  $N_c - 1$ ) of the curved path. Next the 3D entries of the points on the discretized curve  $\mathbf{r}_m^c$ ,  $m = 1 - N_c$  are computed also using the parameters describing  $Z(x, y)$  (see eqs. 7-9). The quantities  $\mathbf{r}_m^c$  as well as  $y_p'^c$ ,  $p = 1 - N_c$  and the array with distances between the discretization points (in 3D),

- $\Delta s_m^c$ ,  $m = 1 - N_c$ , are stored for later use in the computations of among other things the *extra time*,  $\tau$ .
3. Next the glance off speed along the curved part,  $v_x$ , is evaluated using eqs. 13-27,  $\mathbf{r}_p^c$ ,  $p = 1 - N_c$ , and the parameters of the road layout (see section II.c). The quantities  $v_{x,m}$  and  $a_{g\parallel v,m}^c$ ,  $m = 1 - N_c$  (see section II.c) are stored for later use.
  4. Similarly to the above the values  $v_{p,m}$  (the pedalling limit) are computed and stored (for the curved section only; see section II.c).
  5. A discretization for the trajectories and other relevant quantities for the entrance section as well as for the exit section are introduced. The coordinates of the discretization points of the trajectory in the entrance section, of which the projection in the  $z = 0$  plane is a straight line parallel to that of the road sides, are computed and stored, say in  $\mathbf{r}_m^b$ ,  $m = 1 - N_b$ . The corresponding step sizes  $\Delta s_m^b$ ,  $m = 1 - N_b$  (chosen to be around the step sizes  $\Delta s_m^c$ , for more or less uniform inaccuracy), should also be stored. Its first point,  $\mathbf{r}_1^b$ , will be located on the starting line (see Fig. 1). Its last point should coincide with  $\mathbf{r}_1^c$ . Similarly the discretization of the trajectory in the end section is introduced and computed, with coordinates  $\mathbf{r}_m^e$ ,  $m = 1 : N_e$  and step sizes  $\Delta s_m^e$ ,  $m = 1 - N_b$ . Also for later use also the gravitational acceleration felt by the cyclist is computed and stored in  $a_{g\parallel v,m}^b$ ,  $m = 1 : N_b$  and  $a_{g\parallel v,m}^e$ ,  $m = 1 : N_e$ .
  6. From the values for  $v_{x,m}$ ,  $m = 2 : N_c - 1$  (the glance off speed) one can construct the so-called safe speed,  $V_{x,m}$  for the same trajectory part, as explained in full detail in section II.c. So, this step ends with a fully defined safe speed array,  $V_{x,m}$ ,  $m = 2 : N_c - 1$  of which the values are partly lower than that of  $v_x$  and partly coincide with these.
  7. If both  $V_{x,1} > v_s$  and  $\mathbf{r}_1^c$  lies before the transition element the forward calculation can start (see item 8). If not the above *backwards calculations*, but now for the speed values  $V_x$  have to be continued into the lhs straight section with discretization points  $\mathbf{r}_m^b$ ,  $m = 1 - N_b$  mentioned above (see also section II.c). The newly computed values for the safe speed,  $V_{x,m}^b$ , should be stored for use of the (forward) computations.
  8. Now we can start the computation of relevant quantities for the cyclist (moving forward!) starting at point  $\mathbf{r}_1^b$  with a speed  $v_1^b = \min(v_s, V_{x,1}^b)$ . These quantities are evaluated using steps given by eqs. 28-31 with an acceleration according to eq. 11, but with gravitational acceleration values calculated earlier ( $a_{g\parallel v}^b$ ,  $a_{g\parallel v}^c$ ,  $a_{g\parallel v}^e$ ) and assuming zero braking power, reasoned as follows. Instead of accounting for braking power in a direct way it is checked whether the speed is not higher than the corresponding safe speed value ( $V_x$ ); if so it is put equal to that value. From the above backwards calculation we know that this way we always obtain correct (safe) results. If required the applied braking power (in fact, deceleration) can be derived afterwards from other parameters, as showed in section III. While evaluating things in the curved section it is checked in each step if the local pedalling limit velocity,  $v_p$ , is not exceeded. If so, the input power delivered by the cyclist should drop to zero. During all these steps the elapsed time is added up taking into account retardation if the track is off road, as explained in the text around eq. 35; also the speed in each point is stored.

9. The starting speed or the end speed may be smaller than  $v_s$  which, if so, would also cause an excess loss in travelling time. If this happens to be so one should compute the excess time needed to slow down to the required starting speed and/or that for speeding up to the stationary speed  $v_s$ . This excess time can most conveniently be determined by creating, before the optimization process starts, two pairs of arrays (for begin and end sections). The pair for the entrance section, for example, consists of an array with possible starting speeds in a (wide) range of e.g.  $\approx 0.1v_s - v_s$  and the corresponding *extra time* (real time compared to time needed with a speed  $v_s$  for the same trajectory) which is needed to attain the required speed  $V_{x,1}^b$ , starting with a speed  $v_s$  using full braking power. These arrays can be computed by the ‘backwards computations’ mentioned under point 5 above. Note that no detailed structural information (apart from the slope) is needed as the above concerns straight paths on a uniform part of the road. The two arrays for the exit section are made similarly. Then, if needed, the required excess time(s) can easily be computed via interpolation.

For optimization of the trajectory the module for the extra time computation can be called by an optimization routine. We have used, for the results presented below, the minimization routine ‘fminsearch’ from Matlab, which requires starting values for parameters to be optimized (here the polynomial coefficients) as an input and returns optimized values. We have found that the result may depend on the choice of the starting parameters with, apparently, as solutions *local* (as to parameter space) optima. This suggests that, on optimizing, one should try different (reasonable) sets of starting parameters. One could for example start with polynomials of degree 2 and increase the degree by 1 a number of times after each successful optimization step using the resulting coefficients plus one extra coefficient for the higher order term, which may initially be put equal to zero. It is recommendable (to ease the optimization process) to choose the coefficients such that a considerable part of the polynomial curve stays (rather) well within the road borders.

### **III. Computational results**

As an illustration to the above scheme a situation is studied corresponding to the parameters given in fig. 3. These are chosen to be more or less common, including a slope of -6% (which means the cyclist goes indeed downhill with a slope angle  $\varphi = \tan^{-1}(0.06) = 3.43^\circ$  in the straight sections), and also suitable to serve as a starting point to study the effect of ascending outer sides in bends and the dropper seat post, to be discussed at the end of this section.

As an input to the computations we often tried different sets of polynomial coefficients, and have found that the result may in some cases depend strongly on that input, which implies that the optimization routine converges often to local (in parameter space) minima of the extra time,  $\tau$ . The best results (lowest  $\tau$ ) for the situation discussed below were obtained using coefficients corresponding to the optimum fit of a data set describing a curve, with radius, say,  $R_{\text{par}}$ , parallel to the road sides in the circular part with small, straight continuations thereof, also parallel to the road sides (all projected onto the  $z = 0$  plane), into the two straight parts. The exact assumed value of  $R_{\text{par}}$  sometimes (but not always as we will see) hardly played a role in the outcome. However, in cases that the road is ascending to the outer side of the bend, as discussed hereafter, it is required that the trajectory of the starting parameters indeed ‘feels’ that slope. The extra travel time corresponding to considered case, with parameters as given in fig. 3 and an order of 3-4 for the polynomial degree (using higher orders lead to the same results) is 4.30 seconds.

In general, in this figure and figures to come the blue and red dots in this figure indicate the start and end points of the region where the brake has been used, the magenta dots mark the region(s) where pedalling may be forbidden (more details can be inferred from figures below). It is seen from the figure that the optimum trajectory is along the inner bend side, and its projection in the  $z=0$  plane is nearly symmetric with small entries in the odd coefficients ( $A_1, A_3$  etc.) of the polynomial (see eq. 10).

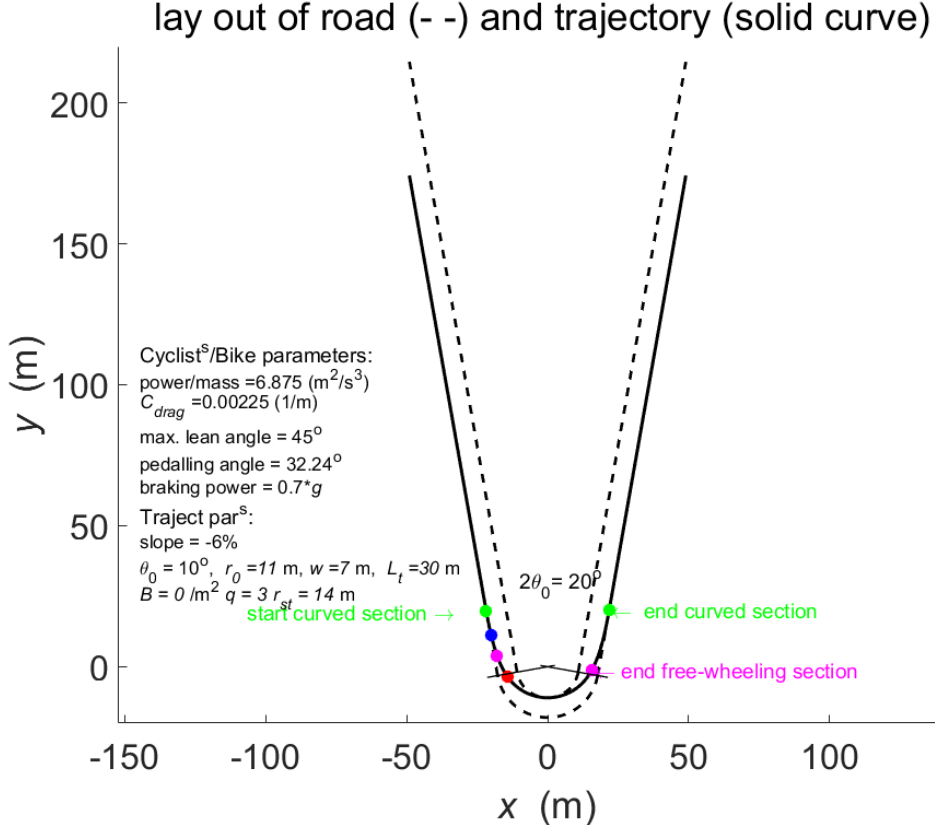


Figure 3. Layout of the bended road and optimized trajectory projected onto the  $z=0$  plane, assuming parameter values as given inside the picture. The green dots indicate the region with a curved trajectory. In the straight road sections the trajectory is virtually coinciding with the outer road sides.

More information on the actions of the cyclist on his way along the bend can be inferred from Figs. 4 and 5, showing speed related issues and the various contributions to the acceleration, respectively.

The region in between the blue and red dots in figures 3 and 4 indicate a region where the wheeler is *fully* (in considered case) braking, in order not to exceed the safe speed,  $V_x$ , a thing that can better be recognized in fig. 5. It is also seen from that figure that it takes quite a distance to get back to the stationary speed,  $v_s$ , which occurs outside the window (but is accounted for in the extra time of  $\tau = 4.30$  s).

The acceleration curves in fig. 5 have been obtained from the local slope along the cyclists path for the gravitational acceleration (see eq. 5, blue curve in the figure), the computed speed,  $v$ , for the drag effect of air (see eq. 4; magenta curve) and changes of the speed for the total acceleration (black pluses) using finite differencing on the basis of eq. 32. The differences between the latter and the sum of the other two is presented by the cyan curve representing the rest:  $a_p + a_b$  (see eqs. 3 and 6), being the acceleration due to input power ( $a_p > 0$ ) and deceleration due to the use of the brake ( $a_b < 0$ ). In Fig. 6 it is seen that the cyan curve can be

positive (input by the cyclist and no brake, of course), negative (use of brake) or zero (no brake, no pedalling).

The green, dashed line in the figure, presenting the minimum acceleration (no input power, full brake) at given speed, is evaluated with eq. 34. If that curve coincides with that of the total acceleration (+++) this corresponds to a situation with full braking power. The latter implies that in such a region the cyan curve takes on the value  $-0.7g = -6.87 \text{ m/s}^2$  (see fig. 3 for the value of the braking power). The above is seen in fig. 5 in the region  $-20 \text{ m} \leq x \leq -14 \text{ m}$ .

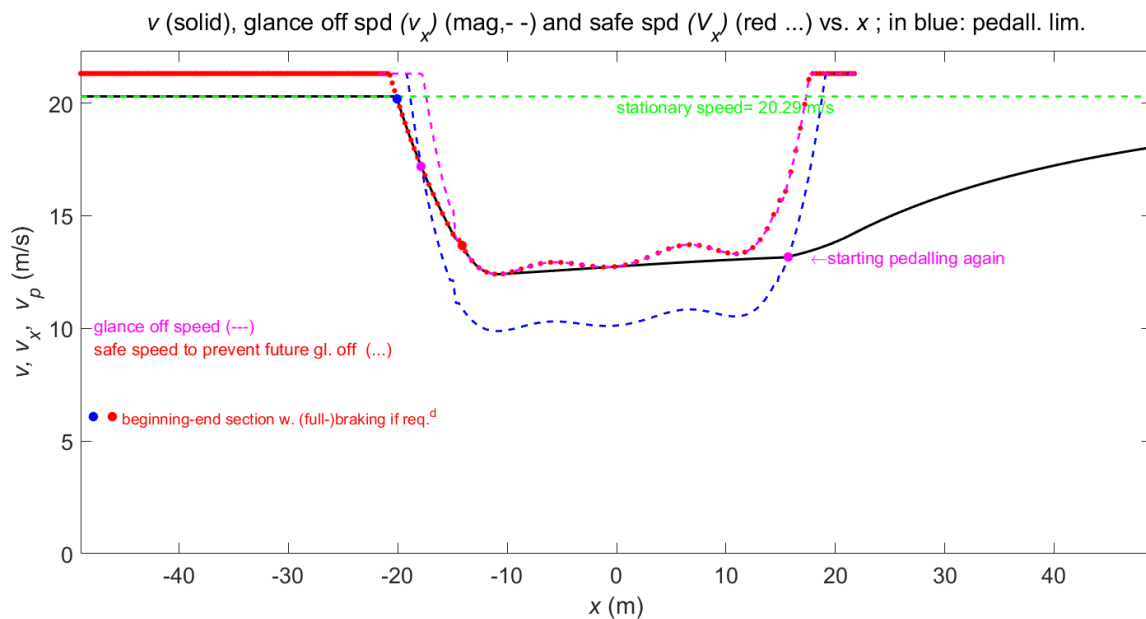


Figure 4. Plot of three speed related quantities ( $v, v_x, V_x, v_p$ ) vs.  $x$  (see also text) for  $B=0$

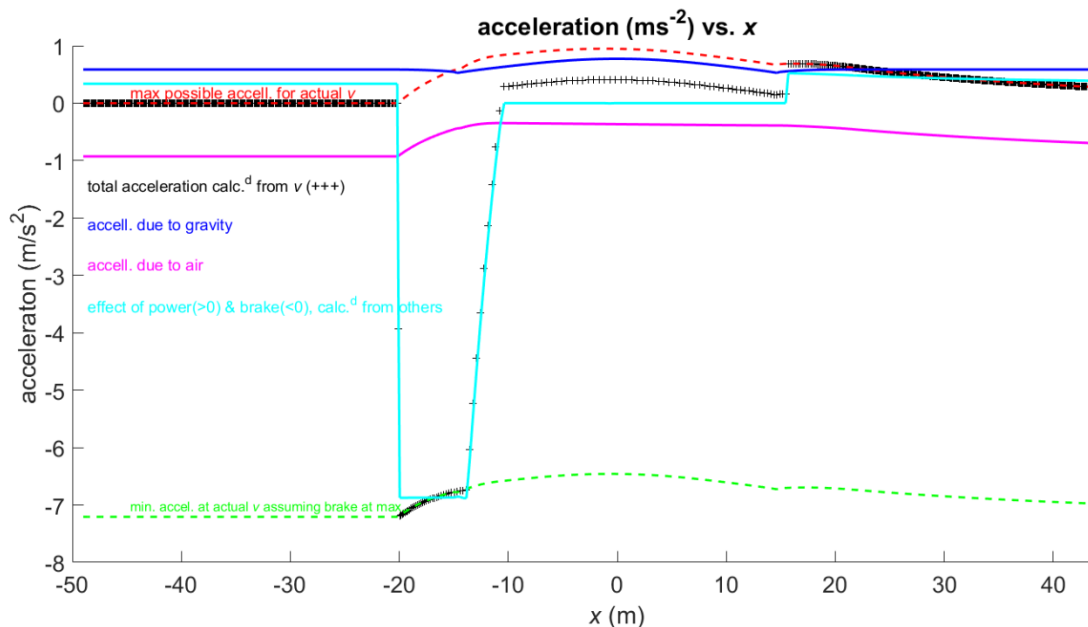


Figure 5. Plot of accelerations of the cyclist vs.  $x$ , showing he effects of all his actions (see also main text).

If the plusses coincide with the red dashed line (maximum possible acceleration at given speed) the input power is at the cyclist's maximum.

As a final picture to this numerical experiment we show the curves with relevant lean and slope angles along the curved trajectory, in fig. 6. It is seen that everywhere the lean angle remains nicely below  $45^\circ$ .

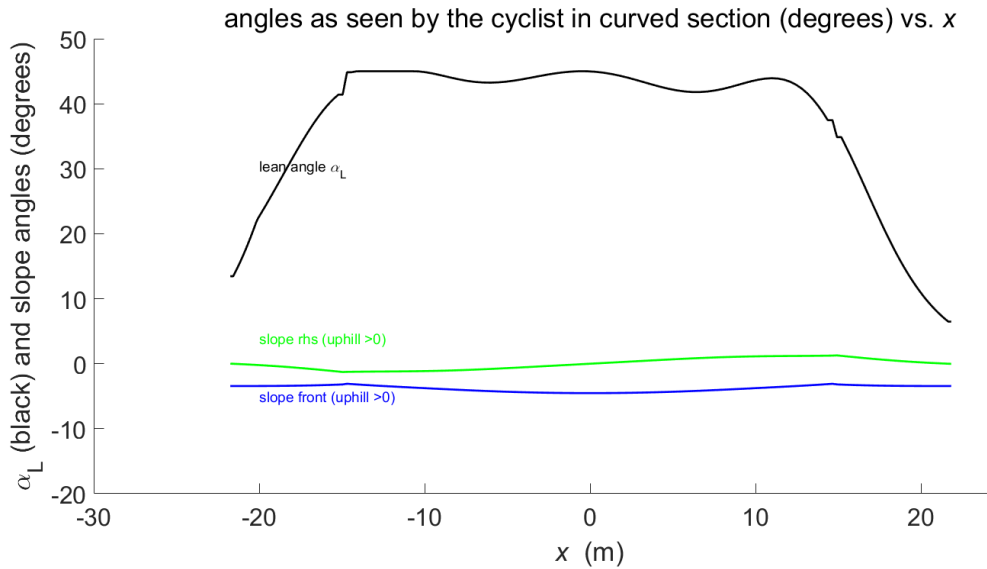


Figure 6. The lean angle  $\alpha_L$  and the slope angles in front of the cyclist (blue) and at his rhs (green) vs.  $x$ .

Next, the effect of a road surface ascending to the outer bend via the parameters  $B$ ,  $r_{st}$  and  $q$ , introduced in section II.b will be discussed. We chose the values  $q = 3$  and  $r_{st} = 3$  m;  $B$  is varied. In fig. 7 we have plotted optimized values of  $\tau$  vs.  $B$ , using different starting values of the polynomial coefficients  $A_{1:7}$ . These coefficients were computed as mentioned at the beginning of this section using different values of  $R_{par}$  (also introduced at the beginning of sect. III), namely  $\sim 13.8$  m and  $\sim 17.46$  m, being 2.8 m from the lhs and  $\sim 0.5$  m from the rhs of the road, respectively.

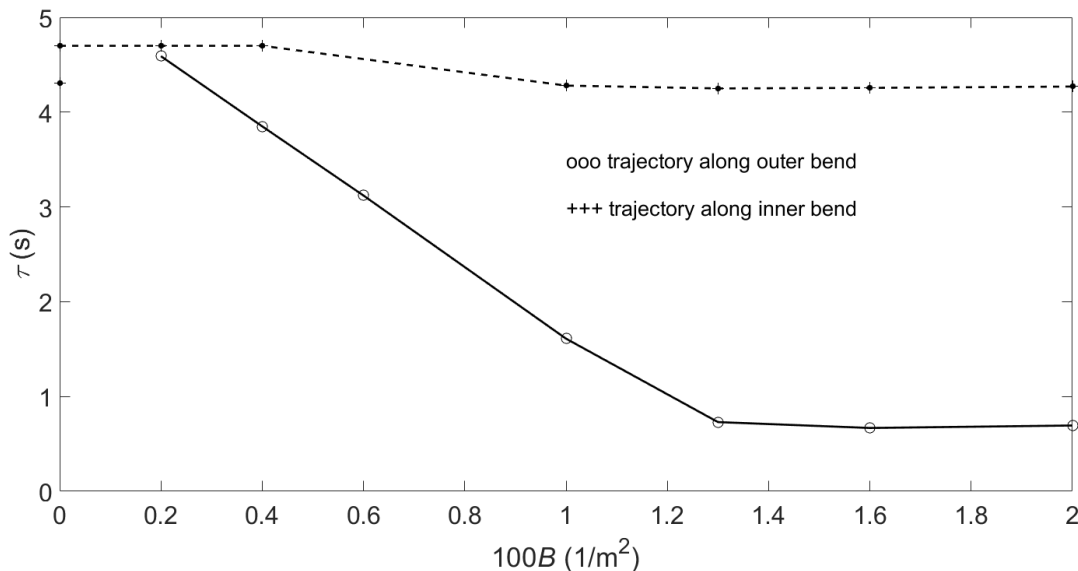


Figure 7. Plot of extra time  $\tau$  vs.  $B$  using different starting values of the polynomial coefficients  $A_{1:7}$  used in the optimization search.

Using the former value, corresponding to a trajectory for which the ascending rhs of the road is not ‘felt’ yet, leads to an optimum (which is, so to see, a local optimum for  $B \gtrsim 3 \cdot 10^{-3} / \text{m}^2$ ) in

which the trajectory runs close to the lhs of the inner bend (as e.g. in fig. 3). The latter value of  $R_{\text{par}}$  leads for considered cases, except the one with  $B = 0$ , to a trajectory very close to the rhs of the outer bend (an illustration of this will be given in fig. 8). It is clear from the figure that with the larger starting parameter ( $R_{\text{par}} \approx 17.5 \text{ m}$ ) a considerable lowering of the optimized travel time occurs for  $B \gtrsim 5 \cdot 10^{-3} / \text{m}^2$ . This can be understood if one takes into account that for example a value  $B \approx 2 \cdot 10^{-2} \text{ m}^{-2}$  corresponds to an elevation of  $\sim 1.3 \text{ m}$  and a local slope of  $\sim 45^\circ$  at the extreme rhs, implying highly improved glance off conditions. At values of  $B \approx 2 \cdot 10^{-3} \text{ m}^{-2}$  the exact position of the (optimized) trajectory hardly influences the resulting value of  $\tau$  as is also seen in fig. 7. Next, we will show graphical results for the case  $B = 2 \cdot 10^{-2} \text{ m}^{-2}$  and add a few comments to these.

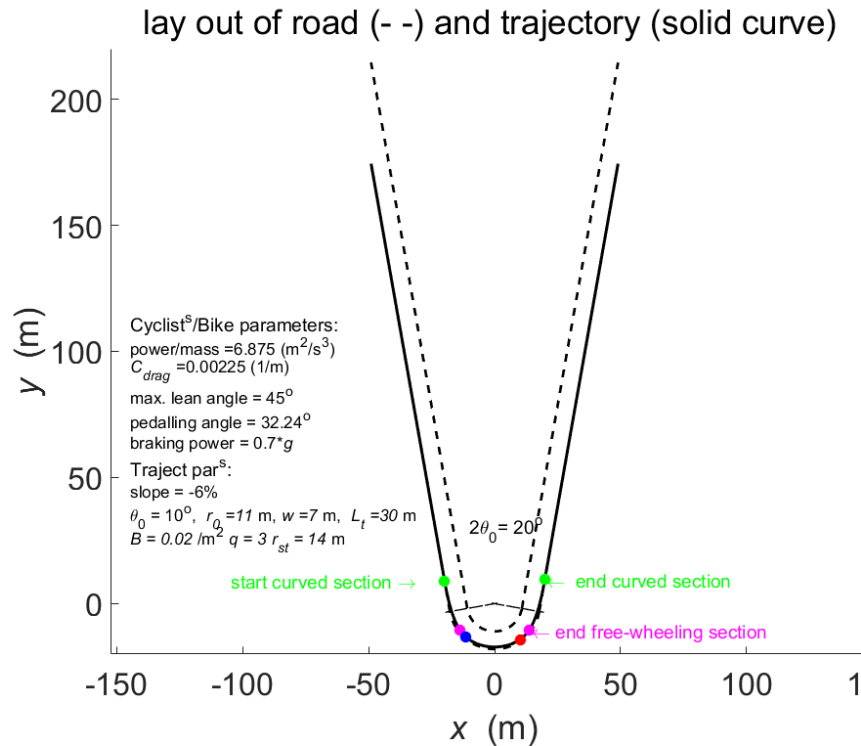


Figure 8. Layout of road and optimized trajectory projected onto the  $z=0$  plane, with an ascending road surface in the bend section with  $B=0.02/\text{m}^2$ . In the straight road sections the trajectory is virtually coinciding with the outer road sides.

In fig. 8 it is seen that the optimized trajectory indeed runs along the extreme outer bend side where the slope in a direction perpendicular to the main road direction is close to  $45^\circ$ . It is noted that its projection onto the  $z = 0$  plane is (again) virtually symmetric. In fig. 9 it is shown how the ascending road surface in the outer bend affects various parameters of which the increase of the safe speed (compared to that given in fig. 4),  $v_x$ , is the most relevant for the low value of  $\tau$ , being equal to  $0.69 \text{ s}$ , as it enables a speed,  $v$ , only a bit lower than the stationary speed,  $v_s$  (green dashed line in Fig. 9).

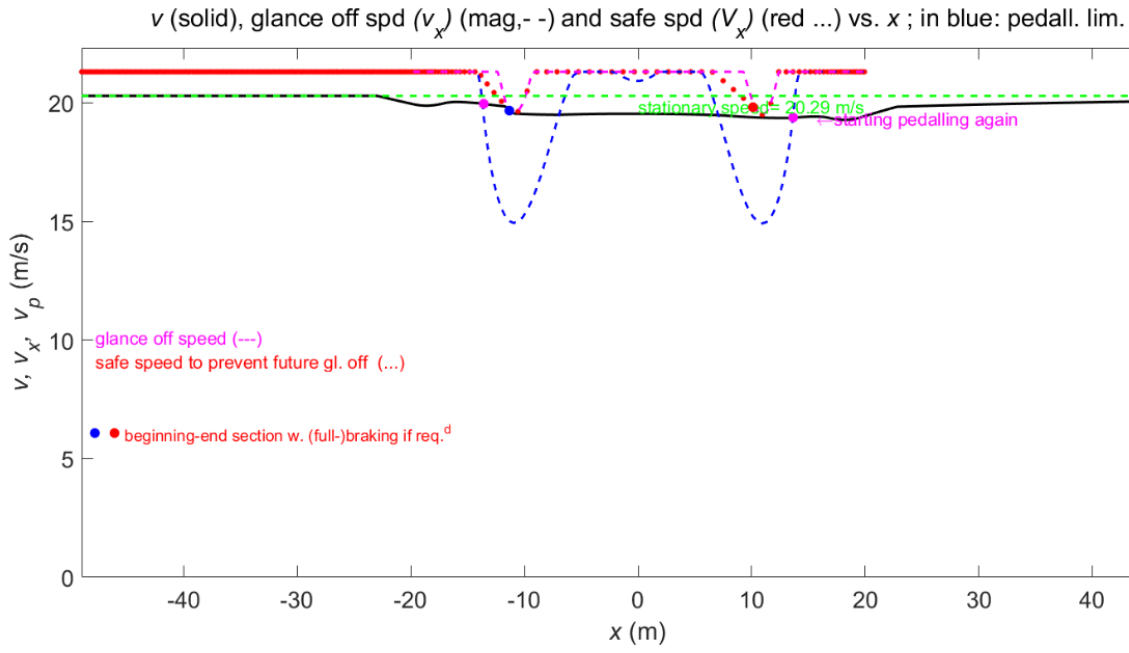


Figure 9. Plot of four speed related quantities ( $v, v_x, V_x, v_p$ ) vs.  $x$  for  $B=0.02/m^2$ . Note that the brake has been used only once along a small distance, which can also and easier be seen in the next figure.

Figure 10 shows curves for all accelerations relevant for presented trip downhill. It is seen that the brake is hardly used in agreement with the short travel time. The sharp features in the blue curve (reflecting the effect of the slope in the cycling direction) are due to entering or leaving the bended area (at  $x = \pm 17.2$  m).

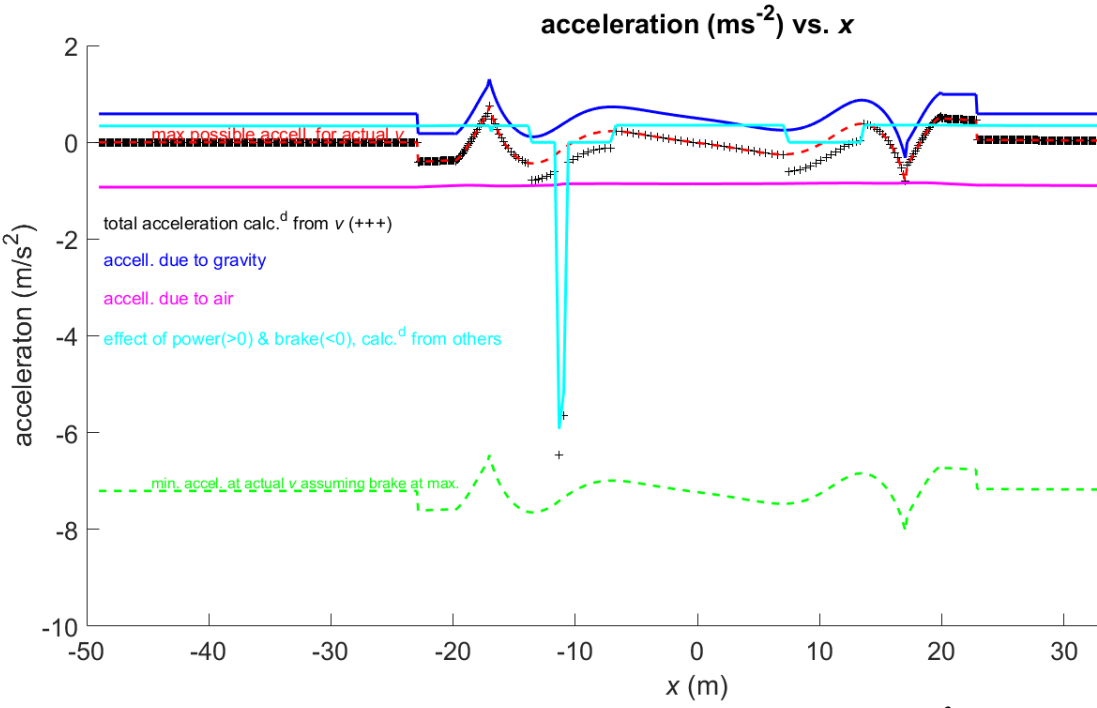


Figure 10. Depicting relevant accelerations for the cyclist's descent for  $B=0.02/m^2$ , with indeed only a small piece where the brake has been used (cyan curve).

We end this part with a 3D picture of trajectory and road structure, which is almost self-explaining with the above in mind (fig. 11), and curves for the lean and slope angles (fig. 12).

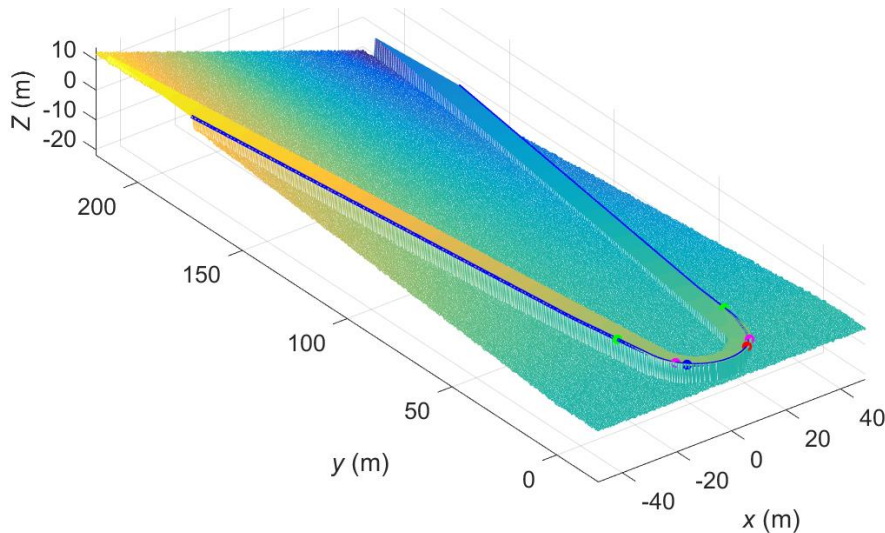


Figure 11. 3D picture of trajectory and road structure for  $B=0.02/m^2$   
 angles as seen by the cyclist in curved section (degrees) vs.  $x$

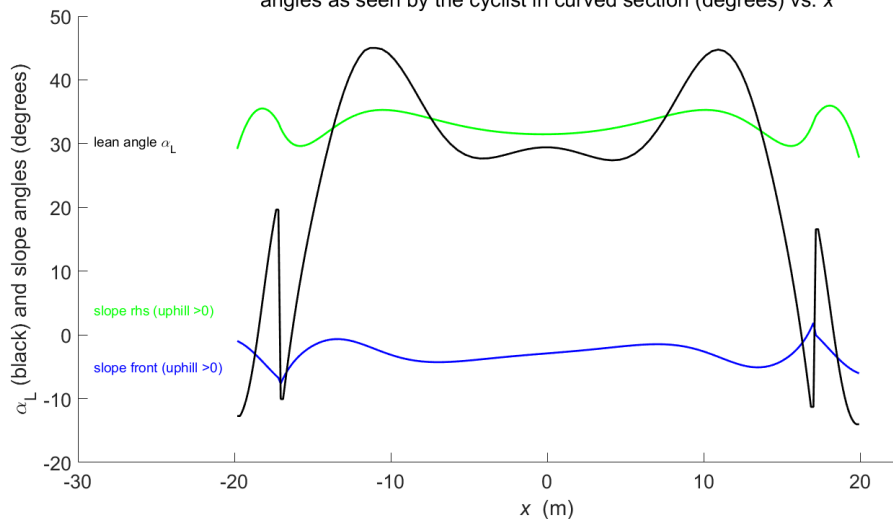


Figure 12. Lean angle and slope angles relevant for the optimized trajectory for  $B=0.02/m^2$

The sharp features in the black curve in fig. 12 are due to entering or leaving the bended area (at  $x = \pm 17.2$  m).

Now we may consider the rapid descent of the Passa di Ganda by Fausto Masnada in the Tour of Lombardy in 2021 [2], where he won half a minute on his competitors, in the light of the above. Taking into account that his competitors (mostly, if not all) chose a non-optimal trajectory along the inner bend sides (like the one depicted in fig. 3), whereas Masnada –who apparently knew this mountainous trajectory very well- took the extreme outer bend sides in the twisty descent from the Passa di Ganda (as is seen in figs. 8 and 11),. If in one such a bend one can gain  $\sim 3$  s this way – as we have shown above (see fig. 7) - one may guess that at least a great part of his time gain of  $\sim 30$  s came from his descending technique. For more definite statements in this matter more study and a better knowledge of the road layout would be required.

A final item we will consider is that of the dropper seat post, as used by Matej Mohoric during his descent of the Poggio in Milano-San Remo, 2022. It is mentioned [3] that a lowering of 6 cm can be achieved, which implies, assuming a lean angle of  $\alpha_L = 45^\circ$  in bends, an increase of the radius of trajectory in the bend of  $6/\sqrt{2}$  cm  $\approx 4$  cm . Assuming the same parameters as used

for fig. 3 above, i.e., without an ascending road surface to the outer side, leads to a road structure similar to that of the first bend after the top of the Poggio, along the Via Grossi Bianchi [3,4]. Our computations lead to a lowering of only 0.01 s for  $\tau$  (from 4.31 s to 4.30 s), which corresponds, with  $v_s = 20.3$  m/s, to an extra distance of only ~20 cm. Considering the movie on [5] it seems that he gained around 1 – 2 m in each bend. So, we may conclude that the dropper seat post contributed hardly to the victory of Matej Mohoric in that tour, even if we take into account that there are some 5 to 6 bends, including a few hair pin bends (see also [4,5]). A minor effect of being only a bit larger distance ahead of competitors may possibly be caused by the advantage that then these people have less profit from the slipstream of the leading cyclist, and that, in addition, the leader will have more profit from the always present motor cycles. But, it may be obvious that his victory was not exclusively due to that dropper seat post.

### **References**

- [1] [https://en.wikipedia.org/wiki/Cubic\\_equation](https://en.wikipedia.org/wiki/Cubic_equation)
- [2] <https://www.touretappe.nl/ronde-van-lombardije-2021/>
- [3] <https://www.cyclingnews.com/news/mohoric-ive-destroyed-cycling-now-everyone-will-use-a-dropper-post/>
- [4] <https://www.touretappe.nl/milaan-san-remo-2022/>
- [5] <https://www.google.com/maps/place/Via+Grossi+Bianchi,+18038+Sanremo+IM,+Itali%C3%AB/@43.8276997,7.8147545,404m/data=!3m1!1e3!4m5!3m4!1s0x12d27537b98887fd:0xc76923ca2e8d3a64!8m2!3d43.8325542!4d7.8121735>
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